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A Method for Demonstrating Relationships in Four-Dimensional Figures

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A method has been devised whereby two three-dimensional projections of a four-dimensional figure can be viewed stereoscopically to permit direct perception of the fourth dimension. The necessary concepts are introduced by a consideration of stereoscopy in two and three dimensions. It is shown that an extra degree of freedom is introduced in the stereoscopic viewing of three-dimensional stereo pairs which permits a direct experimental demonstration of the relationship between right-handed and left-handed three-dimensional figures in terms of a rotation in four-dimensional space.

In a recent paper (Whittaker, 1973) it has been shown that interesting models can be made using three-dimensional projections of four-dimensional figures, if a continuously variable colour parameter is introduced to represent the fourth coordinate of each point of the figure perpendicular to the three dimensions of the projection. Various projections of the four-dimensional hyper-cube were constructed with colour perspective of this kind to illustrate the symmetry of the hyper-cube, and the relationships between some of these projections suggest that a suitably chosen pair of projections might profitably be viewed stereoscopically.

1. The two-dimensional analogue

The concepts involved are conveniently introduced by considering the problem of making our third dimension visible to a 'two-dimensional man' equipped with binocular vision. Fig. 1(*b*) shows a two-dimensional projection of eight spheres located at the vertices of a cube in a general orientation. The two points near the centre represent the vertices of the cube that are nearest to and farthest from a three-dimensional observer, and the figure is readily interpretable in this sense. To a two-dimensional observer in the plane of the paper, however, these two points will be seen by binocular vision to be within the hexagon outlined by the other six. His two eyes, and two different monocular views of Fig. 1(*b*), are shown in Fig. 1(*a*), and his view of Fig. 1(*b*) in its own plane may be seen by viewing Fig. 1(*a*) as a stereo pair. The two images in Fig. 2(*b*) are projections of the cube in directions a few degrees apart, which together constitute a stereo pair. A three-dimensional observer, viewing one with each eye in the usual way, sees the cube, outlined by its vertices, in three dimensions. A two-dimensional observer, viewing one member of the pair with each eye, will see two one-dimensional projections of them, as at Fig. 2(*a*). Binocular fusion of these two images shows that the two central points are indeed those nearest to and furthest from the observer. What one sees is a two-dimensional projection of the same cube but on a plane perpendicular to the page. What the two-dimensional observer sees is a picture in his own two-dimensional world in which distance along his line of sight has been suppressed, and our third dimension (of which he previously knew nothing) has taken its place.

2. The experimental device

This scheme can be adapted directly to permit a fourth dimension to be viewed in place of one of our three dimensions. Three-dimensional models of two projections (with slightly different projection directions) of a four-dimensional figure have to be viewed in the same orientation, one by each eye. Perception of the line of sight dimension is then suppressed, and replaced by the mean projection direction. This procedure has been realized with the use of models of two general orthographic projections of a four-dimensional hyper-cube projected in directions 3° apart. The two models standing side by side are shown in ordinary stereoscopy in Fig. 3(*a*). Because the differences between the two models are very small they have to be constructed to a high degree of (relative) accuracy. In order to minimize the difficulty of construction they were accordingly made large, about 30 cm in diameter, and viewed with a large reflecting stereoscope (with its eye lenses removed) at a distance of about 400 cm. It is important that the line of sight to each model should be perpendicular to the inter-ocular direction for which the models are designed, and this is most easily achieved if the models are aligned so that this direction in each of them is collinear and they are separated by a distance equal to the distance between the mirrors of the stereoscope.

It is possible to get an idea of the performance of the device from Fig. 3(*a*). If we denote the left and right models by L and R, and the left and right eyes' views of a model by l and r, then the four components of Fig. 3(*a*) are Ll, Rl, Lr and Rr, reading from left to right. Normal viewing of Fig. 3(*a*) stereoscopically fuses the perceptions of Ll with Lr and Rl with Rr to give an ordinary stereoscopic view of the two models side by side. However, if suitable arrangements are made to fuse Ll with Rr (ignoring Rl and Lr) one sees the effect of using the device itself. To facilitate this *Ll* and *Rr* are reproduced at a standard separation in Fig. 3(*b*). When viewed in this way it may be noted in particular that the two intersections nearest the centres of the photographs of the models appear at the front and back, whereas in the actual models they are near the centre of a 'cage' formed by the other 14 vertices, as may be verified by normal stereoscopic viewing of Fig. 3(*a*). This corresponds exactly to the demonstration to a two-dimensional observer that the two points

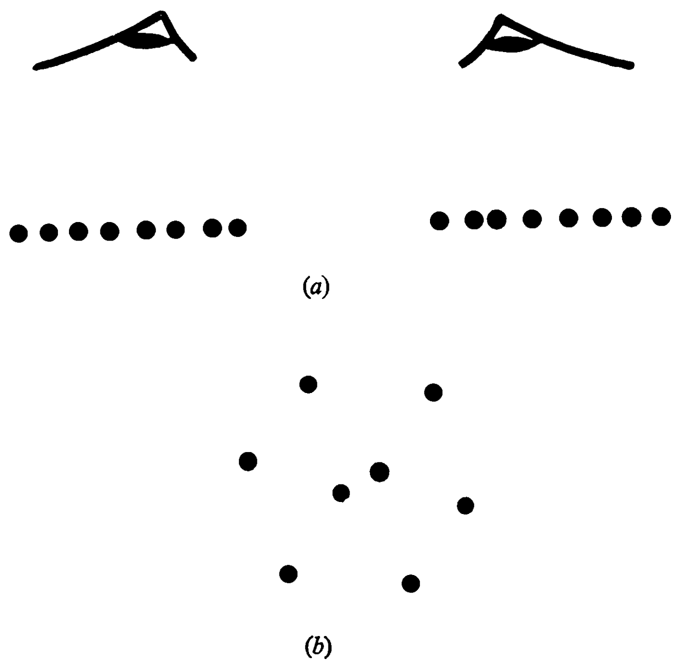


Fig. 1. (*a*) Two one-dimensional projections of (*b*). (*b*) Two-dimensional projection of the vertices of a cube. Stereoscopic viewing of (*a*) gives a stereoscopic view of (*b*) as would be seen by a two-dimensional observer in its own plane.

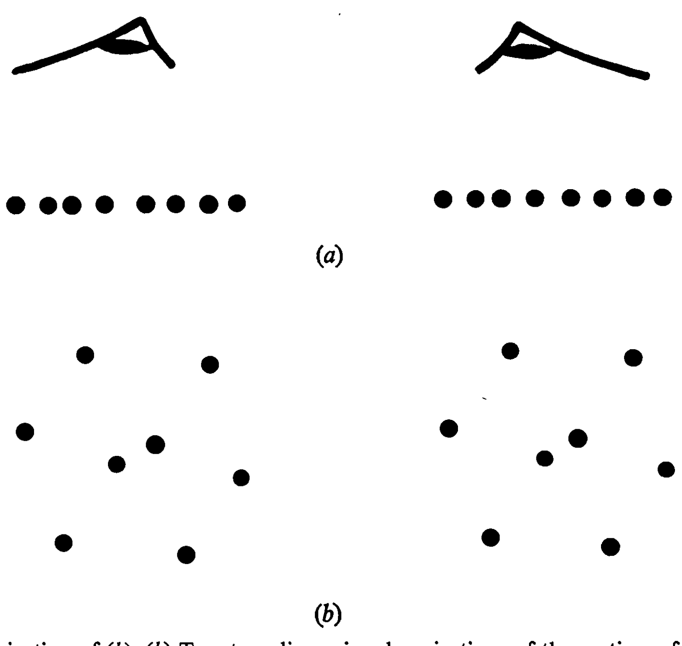


Fig. 2. (*a*) One-dimensional projection of (*b*). (*b*) Two two-dimensional projections of the vertices of a cube arranged as a stereo pair. Stereoscopic viewing of (*a*) gives a stereoscopic view, in a plane perpendicular to the paper, of a projection of the cube on that plane. This is equivalent to the two-dimensional observer's views of (*b*) by binocular fusion of (*a*).

near the centre of Fig. 1(*b*) can be meaningfully regarded as 'above' and 'below' the other six in a third dimension.

In the use of the device described above each eye sees a two-dimensional projection of one of the two models, and it is therefore possible to reproduce what is seen in the experiment by stereoscopic viewing of the stereo pair of two-dimensional images in Fig. 3(*b*). It might therefore appear that nothing new had been achieved. Instead of the pair of models shown in Fig. 3(*a*), a single model could have been built which could have been photographed stereoscopically, and the resulting stereo pair would have been identical with Fig. 3(*b*) in all respects. There are however some interesting features in the stereoscopic viewing of a pair of three-dimensional models which do not arise in the case of two-dimensional stereo pairs, as may be shown by the following analysis.

3. Degrees of freedom in stereoscopic viewing

(*a*) *Two-dimensional stereo pairs in two-dimensional space*

Consider a stereo pair of orthogonal projections of a three-dimensional figure as in Fig. 2(*b*). Let the orthogonal coordinates wxy be chosen in the figure such that the interocular direction of viewing will be parallel to the y axis and the mean projection direction is parallel to the w axis. Let the two projection directions each make an angle α with the w axis. In the two-dimensional space of each member of Fig. 2(*b*) let the Y axis be parallel to the interocular direction and an X axis be perpendicular to it. Then the point (w,x,y) projects to points (X,Y) in the two members of the stereo pair such that

$$X=x$$

and

$$Y=\pm w \sin\alpha + y\cos\alpha\,.$$

Stereoscopic viewing of the two images by the 'two-dimensional man' then leads to perception of the point in his perceptual coordinates X', Y' (where the X' and Y' axes are parallel to the X and Y axes) with $X'=w$, $Y'=y$, corresponding to a projection down the x axis.

If the two projections are viewed with the wrong eyes, then the point appears at $X'=-w$, $Y'=y$. That is the perceived object has suffered a reflection in the Y' axis. It will be the projection of the object parallel to the x axis after reflexion in the xy plane.

If each of the two projections is rotated through an angle φ about its own origin the points (X,Y) move to

$$X=\pm w\sin\alpha\sin\varphi + x\cos\varphi + y\cos\alpha\sin\varphi$$
$$Y=\pm w\sin\alpha\cos\varphi - x\sin\varphi + y\cos\alpha\cos\varphi\,.$$

The stereoscopically perceived coordinates are therefore

$$X'=w\cos\varphi$$
$$Y'=-x\sin\varphi/\cos\alpha + y\cos\varphi\,.$$

If the three-dimensional object had been rotated through an angle φ about the w axis before projection the point (w,x,y) would have moved to $(w, x\cos\varphi + y\sin\varphi,\ -x\sin\varphi + y\cos\varphi)$, and would then have projected to

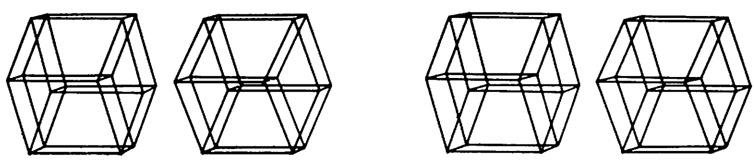
$$X=x\cos\varphi + y\sin\varphi$$
$$Y=\pm w\sin\alpha - x\cos\alpha\sin\varphi + y\cos\alpha\cos\varphi\,.$$

The stereoscopically perceived coordinates would therefore have been

$$X'=w$$
$$Y'=-x\sin\varphi + y\cos\varphi\,.$$

Since α is small the correspondence is very close except for the factor $\cos\varphi$ in X'.

This leads to a complete loss of stereoscopy (giving a one-dimensional figure) at $\varphi=90°$ and $270°$, and



(*a*)

(*b*)

Fig. 3. (*a*) Two three-dimensional models of projections of a hyper-cube, photographed as a stereo pair. (*b*) The left-eye view of the left-hand model and the right-eye view of the right-hand model. When these are viewed stereoscopically the fourth dimension of the hyper-cube, lost in the projection into three dimensions, appears perpendicular to the plane of the paper.

there is an implied reflexion of the three-dimensional object in the xy plane at these positions. Over ranges of φ that do not include these values the effect on the perceived image is approximately equivalent to a rotation of the three-dimensional figure about the w axis, apart from a changing scale factor of $\cos\varphi$ in the X' direction. Rotation through 180° from $\varphi=\varphi_1$ to $\varphi_1+180°$ is equivalent to an inversion of the three-dimensional figure through the origin, but the effect on the perceived image is simply a rotation through 180° in its own perceptual plane.

(*b*) *Two-dimensional stereo pairs in three-dimensional space*

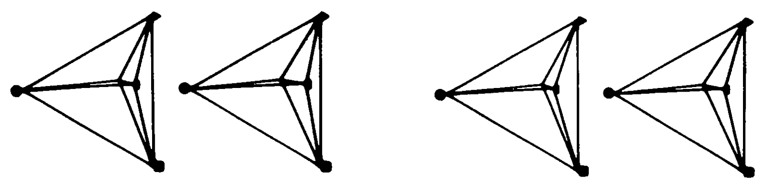
This is the normal way in which stereoscopic diagrams and photographs are used. A point (w,x,y) in the object projects to

$$X=0$$
$$Y=\pm w\sin\alpha+y\cos\alpha$$
$$Z=x$$

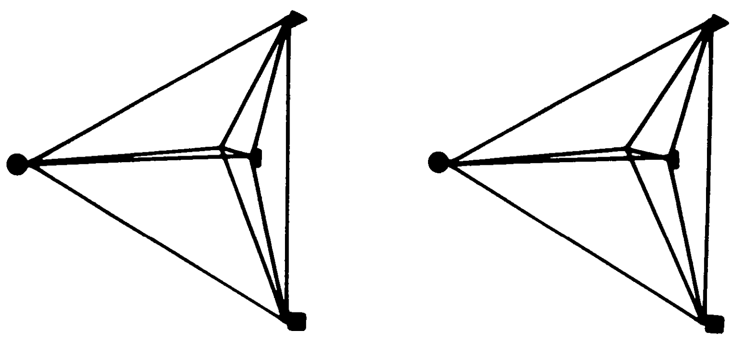
and is perceived in three dimensions at $X'=w$, $Y'=y$, $Z'=x$. (This nomenclature enables us to keep the X' axis as the perceived line of sight, with the Y' axis parallel to the interocular line as before. The Z' axis is perpendicular to both and regarded as vertical.)

The previous analysis holds good for interchange of the two members of the pair, when the effect is a reflexion in the $Y'Z'$ plane.

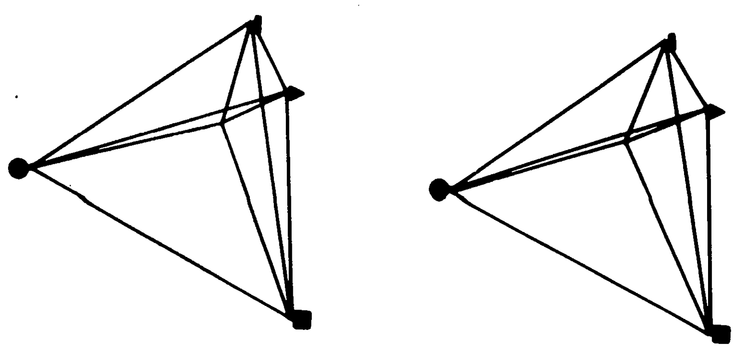
Rotation of the two members of the pair in their own plane about their centres is no longer permissible



(*a*)

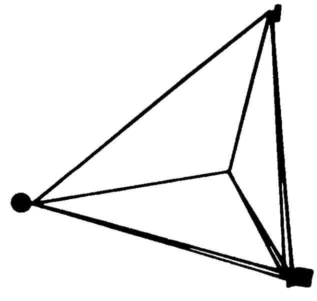


(*b*)

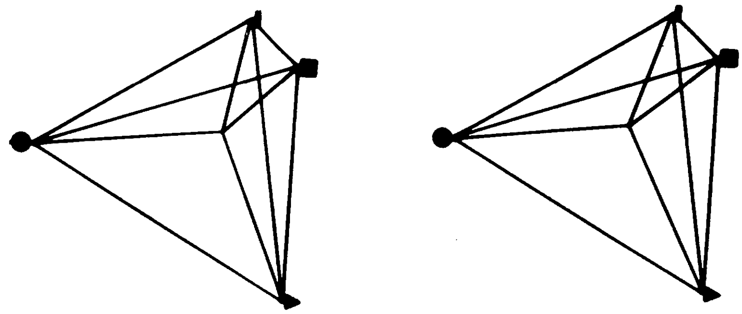


(*c*)

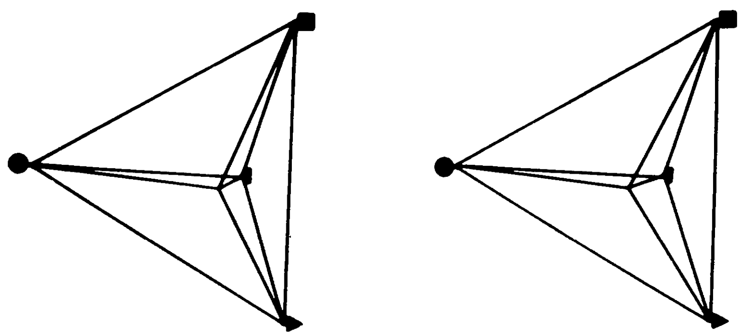
Fig. 4. (*a*) Stereo photograph of two models of three-dimensional projections of a regular pentatope, projected in slightly different directions and having their vertices differently labelled. (*g*) The same after rotation through 180° about a horizontal axis. (*b*) and (*f*) Left eye's view of the left-hand model and right eye's view of the right-hand model of (*a*) and (*g*) respectively. When viewed stereoscopically (*f*) is the mirror image of (*b*) although produced by a rotation. (*c*), (*d*) and (*e*) Intermediate images after rotation by 40°, 90° and 140° from (*b*).



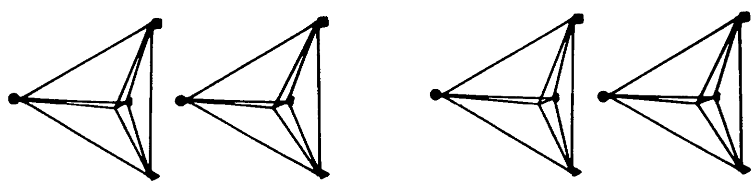
(d)



(e)



(f)



(g)

Fig. 4. (*cont*).

unless $\varphi = 180°$, as it disturbs the parallelism of the y axis and the interocular line. When $\varphi = 180°$ it leads to $X' = -w$, $Y' = -y$, $Z' = -z$, that is to an inversion of the perceived object in the origin.

Provided that the members of the stereo pair are transparent, and we can ignore the practical difficulty of viewing them edge-on, we can consider two further cases, namely rotation about their Y axes or Z axes. Rotation of the two members of the stereo pair about their Z axes leads to a contraction of both X' and Y' to zero after 90°, followed by a re-expansion to the situation where $X' = -w$, $Y' = -y$, $Z' = x$ after 180°

corresponding to a rotation of the perceived image about the Z' axis. Rotation about the Y axes through an angle θ leads to $Z=x\cos\theta$, with Y unchanged. The perceived coordinates are thus $X'=w$, $Y'=y$, $Z=x\cos\theta$, and the image is seen to contract with $\cos\theta$ along the Z' direction, go through a loss of a dimension at $\theta=90°$ and then expand again to the mirror image of the original (reflected in the $X'Y'$ plane) when $\theta=180°$.

(c) Three-dimensional stereo pairs in three-dimensional space

The point (w,x,y,z) of the object is projected to $X=x$, $Y=\pm w\sin\alpha+y\cos\alpha$, $Z=z$, and is perceived at $X'=w$, $Y'=y$, $Z'=z$, *i.e.* as a projection parallel to the x axis.

Interchange of the two members of the pair leads to results identical with those in (*a*) and (*b*). Rotation of the members of the pair about their X axis leads to the same effects as rotation of the members of the pair in their own plane in (*b*); the rotation is limited to a value of 180° and the result is an inversion of the perceived image, and is equivalent to the operation on the four-dimensional figure of a $\tilde{2}$ axis along the x axis.

Rotation of the members of the pair about their Z axes leads to results exactly analogous to those of rotation in (*a*) above. There is complete loss of stereoscopy (giving a two-dimensional figure) at $\varphi=90°$ and 270°, and there is an implied reflexion of the four-dimensional figure in the xyz hyperplane at these positions. Over ranges of φ that do not include these values the effect on the perceived image is approximately equivalent to a rotation of the four-dimensional figure about the wz plane, apart from a changing scale factor of $\cos\varphi$ in the X' direction. Rotation through 180° from $\varphi=\varphi_1$ to $\varphi_1+180°$ is equivalent to the operation on the four-dimensional figure of a $\tilde{2}$ axis along the w axis, but the effect on the perceived image is simply a rotation through 180° in its own perceptual space.

The remaining possibility is rotation of the two members of the pair about their Y axes. After rotation through an angle θ, the projection of the point (w,x,y,z) is at

$$\begin{aligned} X&=x\cos\theta+z\sin\theta\,,\\ Y&=\pm w\sin\alpha+y\cos\alpha\,,\\ Z&=-x\sin\theta+z\cos\theta \end{aligned}$$

and is perceived at

$$X'=w\,,\quad Y'=y\,,\quad Z'=-x\sin\theta+z\cos\theta\,.$$

One therefore sees a continuous succession of different true projections of the four-dimensional figure as θ is varied, the directions of projection taking successively all directions in the xz plane as the figure is rotated about the wy plane. There is no loss of dimensionality at any value of θ but after rotation from $\theta=\theta_1$ to $\theta_1+180°$ the perceived image has suffered a reflexion in the $X'Y'$ plane.

4. Discussion

Two kinds of operation on a stereo pair exist. The first kind is a two-valued operation only (interchange or a rotation inherently limited to 0 or 180°). This always leads to an improper transformation of the perceived image (reflexion or inversion) and to an implied improper transformation of the object. The second kind of operation is a rotation characterized by a continuous angular variable, and therefore leads to a continuous change in the nature of the image. In every case except one it involves a loss in dimensionality of the image at particular points of the rotation, and when there is a loss of one dimension at such a position there is an implied reflexion of the object during passage through that position. The change in the image after rotation of the members of the stereo pair through 180° may be of the nature either of a rotation through 180° or of a reflexion. In the latter case we have a continuous series of images and the two ends of the series are mirror images of one another. However no conceptual difficulty arises because at some position along the series the image has lost one dimension.

In the remaining case, that described last in § 3, the operation corresponds to a rotation of the object; there is no loss of dimensionality at any point during a rotation through a range of 180°; there is a continuous series of images throughout the operation; and yet the image at the end of the series is related by a reflexion to that at the beginning of the series. In effect one has gone 'round the looking-glass' instead of 'through the looking-glass'.

It is of course well known that analytically one can relate a reflexion in n dimensions to a rotation through 180° in $(n+1)$ dimensions, but the device described in § 2 enables one to demonstrate the process physically for $n=3$. It would not be possible to carry out a similar demonstration with $n=2$ for a 'two-dimensional man', because the demonstration requires that the invariant element of the rotation should contain the interocular direction. With $n=2$ this is impossible because the invariant element of a rotation is a point.

Fig. 4 is analogous to Fig. 3 except that it contains non-centrosymmetric figures, namely two projections of a regular pentatope having its vertices distinguished by differently shaped labels. An ordinary stereoscopic view of Fig. 4(*a*) shows the two models side by side, and stereoscopic fusion of the extreme left and right images [reproduced for convenience at (*b*)] shows the effect achieved by the device. Fig. 4(*g*) shows the same models after rotation through 180° about their line of centres, and they can be seen to be of the same hand as in Fig. 4(*a*). However stereoscopic fusion of the extreme left and right images [reproduced at (*f*)] reveals a change of hand. Fig. 4(*c*)–(*e*) show some of the intermediate images produced during the rotation.

Reference

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